Answer model exam "Lie groups in Physics" of November 4, 2015

Points

8 — 1a) dimension of $U(n)$ = dimension of $u(n)$, algebra of $U(n)$

80% {

$U \in U(n)$ if $U^\dagger = U^{-1}$ $\Rightarrow$ $a \in u(n)$ if $a^\dagger = -a$

$\Rightarrow a = \begin{pmatrix} i\alpha_1 & \beta_1 + i\beta_2 & \cdots \\ -\beta_1 + i\beta_2 & \ddots & \vdots \\ \vdots & & i\alpha_n \end{pmatrix}$

independent components on diagonal $= n$

" " above diagonal: $2 \cdot \sum_{i=1}^{n-1} i = 2 \frac{(n-1)\cdot n}{2} = n^2 - n$

}

$\Rightarrow$ dimension of $U(n) = n^2$

20% { dimension of $SU(n)$ = dimension of $U(n) - 1$

since there is 1 constraint: $\det U = 1 \Rightarrow \operatorname{Tr} a = 0$

$\Rightarrow \alpha_n = -\sum_{i=1}^{n-1} \alpha_i$ }

5 — 1b) $SU(n)$ invariant subgr of $U(n)$?

30% { obviously subgp: $\forall U_1, U_2 \in SU(n) \Rightarrow \det(U_1 U_2) = \det U_1 \det U_2 = 1$

$\Rightarrow U_1 U_2 \in SU(n)$ }

70% { invariant: $\forall U \in U(n)$ $\forall U' \in SU(n) \Rightarrow \det(U U' U^{-1}) = \det U' = 1$

$\Rightarrow U U' U^{-1} \in SU(n)$ }

5 — 1c) Show $U(n)/SU(n) \cong U(1)$:

$\forall U \in U(n)$: ~~det~~ $U U^\dagger = 1$

$\Rightarrow \det U U^\dagger = 1 \Rightarrow \det U = e^{i\varphi}$

as $\det U_1 U_2 = \det U_1 \det U_2$

it follows that det is a homomorphism from $U(n) \to U(1)$

ker det $= SU(n)$ $\Rightarrow$ $U(n)/SU(n) \cong U(1)$

7 — 1d) defining reps of $U(n)$ & $SU(n)$ are irreps, so

80% {

Schur's lemma implies $\forall U \in Z(U(n))$ and $\forall U' \in U(n)$

$D_{def}(U)\, D_{def}(U') = D_{def}(U')\, D_{def}(U)$

$\Rightarrow D_{def}(U) = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{C}$.

$\lambda \mathbb{1} \in U(n)$ iff $\lambda^* = \lambda^{-1} \longrightarrow \lambda = e^{i\phi} \Rightarrow Z(U(n)) = U(1)$ }

20% { For $SU(n)$ there is an additional condition $\lambda^n = 1 \Rightarrow Z(SU(n)) = \mathbb{Z}_n$

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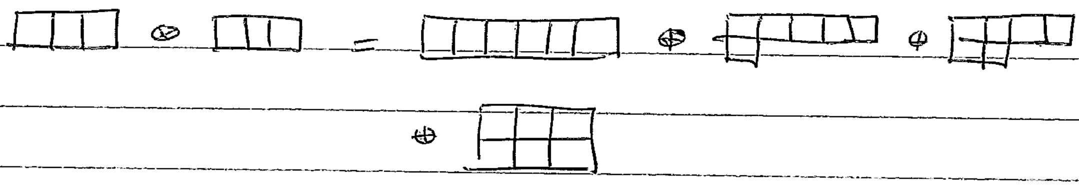
5	1a)	$SU(2)$ → isospin symmetry (approximate) or angular momentum (exact) ; $SU(3)$ → flavor (approx.) or color (exact) — 50%
		$U(1)$ → Electromagnetism (exact), $B-L$, etc (or $SO(2)$ examples) — 50%

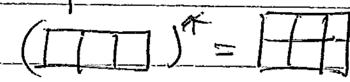
10 2a)



10 2b) 90%: $n=2$: $\underline{4} \otimes \underline{4} = \underline{7} \oplus \underline{5} \oplus \underline{3} \oplus \underline{1}$

$n=3$: $10 \otimes 10 = \underline{28} \oplus \underline{35} \oplus \underline{27} \oplus 10^*$

10%



8 2c) $n=2$ application to addition of angular momentum

$\underline{4} \hat{=} j_t = 3/2$

$j_1 = \frac{3}{2} \times j_2 = \frac{3}{2} \Rightarrow j_3 = |j_1 - j_2|, \ldots, j_1 + j_2$

$= 0, 1, 2, 3$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

dim of irrep: $\underline{1} \quad \underline{3} \quad \underline{5} \quad \underline{7}$.

7 3a) $G = SL(2,\mathbb{C})$ complex 2×2 matrices with determinant 1 act on 2D carrier space without 1D invariant subspaces, hence irreducible. Also, subgp $H = SU(2)$ ~~has~~ has irreducible defining rep, hence extension to G remains irreducible.

(other option: parametrize all elt's of $SL(2,\mathbb{C})$: $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that $\alpha\delta - \gamma\beta = 1$

and show that only matrix that commutes with its is $\propto \mathbb{1}$

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Points

10 3b) $$\exp\left(-\chi\frac{\hat{n}\cdot\vec{\sigma}}{2}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\chi\frac{\hat{n}\cdot\vec{\sigma}}{2}\right)^{n} = \sum_{n=0}^{\infty}\frac{1}{2n!}\left(-\chi\frac{\hat{n}\cdot\vec{\sigma}}{2}\right)^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-\chi\frac{\hat{n}\cdot\vec{\sigma}}{2}\right)^{2n+1}$$

$$(\hat{n}\cdot\vec{\sigma})^2 = n_i n_j(\sigma_i\sigma_j) = n_i n_j\left(\frac{\sigma_i\sigma_j+\sigma_j\sigma_i}{2}\right) = n_i n_j \delta_{ij} = \hat{n}^2 = 1$$

$$\exp\left(-\chi\frac{\hat{n}\cdot\vec{\sigma}}{2}\right) = \sum_{n=0}^{\infty}\frac{1}{2n!}\left(-\frac{\chi}{2}\right)^{2n}\mathbb{1} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-\frac{\chi}{2}\right)^{2n+1}\hat{n}\cdot\vec{\sigma}$$

$$= \cosh\frac{\chi}{2}\,\mathbb{1} + \sinh\left(-\frac{\chi}{2}\right)\hat{n}\cdot\vec{\sigma} = \cosh\frac{\chi}{2}\,\mathbb{1} - \sinh\left(\frac{\chi}{2}\right)\hat{n}\cdot\vec{\sigma}$$

5 3c) 80% $$H(\chi,\hat{n}) \equiv \exp\left(-\chi\frac{\hat{n}\cdot\vec{\sigma}}{2}\right) = \begin{pmatrix} \cosh\frac{\chi}{2} - \sinh\frac{\chi}{2}n_3 & -\sinh\frac{\chi}{2}n_1 + i\sinh\frac{\chi}{2}n_2 \\ -\sinh\frac{\chi}{2}n_1 - i\sinh\frac{\chi}{2}n_2 & \cosh\frac{\chi}{2} + \sinh\frac{\chi}{2}n_3 \end{pmatrix}$$

Clearly in $GL(2,\mathbb{C})$ and $\det H = \cosh^2\frac{\chi}{2} - \sinh^2\frac{\chi}{2}n_3^2 - \sinh^2\frac{\chi}{2}n_1^2 - \sinh^2\frac{\chi}{2}n_2^2 = \cosh^2\frac{\chi}{2} - \sinh^2\frac{\chi}{2} = 1$

Alternatively, $\det\exp\left(-\chi\frac{\hat{n}\cdot\vec{\sigma}}{2}\right) = \exp\operatorname{Tr}\left(-\chi\frac{\hat{n}\cdot\vec{\sigma}}{2}\right) = \exp 0 = 1$

20% H is hermitian, not unitary. $SU(2) \subset SL(2,\mathbb{C})$, so H forms subset

5 3d) $L^{\mu}{}_{\nu}(A) = \frac{1}{2}\operatorname{Tr}(\tilde{\sigma}^{\mu}A\sigma_{\nu}A^{\dagger})$ is 2:1 mapping for $A \in SL(2,\mathbb{C})$

$L^{\mu}{}_{\nu}(A) = L^{\mu}{}_{\nu}(-A)$, so not 1-1

$L^{\mu}{}_{\nu}(\mathbb{1}) = g^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$, hence part of connected subgroup of $O(3,1)$

(uses that $SL(2,\mathbb{C})$ is connected and hence is continuous)

hence not onto $O(3,1)$, but it is onto $L_+^{\uparrow} \cong SO^{\uparrow}(3,1)$.

Alternatively, $L^{0}{}_{0} = \frac{1}{2}\operatorname{Tr}(AA^{\dagger}) > 0$, so $L^{0}{}_{0} \leq -1$ not included.

5 3e) 50-50% H corresponds to a pure Lorentz transformation or boost in the $\hat{n}$-direction, with rapidity χ. Boosts do not form a (sub)group.

$\sum$ points ≤ 90 ~~(Extra due difference of SL(2,C))~~ grade: $\frac{\sum \text{points}}{10} + 1$